

# Free differential Lie Rota-Baxter algebras and Gröbner-Shirshov bases\*

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**Abstract:** We establish the Gröbner-Shirshov bases theory for differential Lie  $\Omega$ -algebras. As an application, we give a linear basis of a free differential Lie Rota-Baxter algebra on a set.

**Key words:** Gröbner-Shirshov basis, Lyndon-Shirshov word, differential Lie Rota-Baxter algebra

**AMS 2000 Subject Classification:** 16S15, 13P10, 16W99, 17A50

## 1 Introduction

Let  $k$  be a field and  $\lambda \in k$ . A differential algebra of weight  $\lambda$  or a  $\lambda$ -differential algebra ([19, 23, 29]) is a  $k$ -algebra  $(R, \cdot)$  together with a differential operator (of weight  $\lambda$ )  $D : R \rightarrow R$  satisfying

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y) + \lambda D(x) \cdot D(y), x, y \in R.$$

The differential algebras were first studied by J.F. Ritt [29] and have developed to be an important branch of mathematics in both theory and applications (see for instance [15, 19, 33]).

A Rota-Baxter algebra of weight  $\lambda$  or  $\lambda$ -Rota-Baxter algebra ([4, 22, 30]) is a  $k$ -algebra  $(R, \cdot)$  together with a Rota-Baxter operator (of weight  $\lambda$ )  $P : R \rightarrow R$  satisfying

$$P(x) \cdot P(y) = P(x \cdot P(y)) + P(P(x) \cdot y) + \lambda P(x \cdot y), x, y \in R.$$

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\*Supported by the NNSF of China (11171118, 11571121) and the NSF of Guangdong (2015A030310502).

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The Rota-Baxter operator on an associative algebra initially appeared in probability [4] and then in combinatorics [30] and quantum field theory [14]. There are a number of studies on associative Rota-Baxter algebras on both commutative and noncommutative case. For more details we refer the reader to [22] and the references given there. The Rota-Baxter operator of weight 0 on a Lie algebra is also called the operator form of the classical Yang-Baxter equation [31]. The Lie Rota-Baxter algebras are closely related with the pre-Lie algebras. Recently, there are many results on Lie Rota-Baxter algebras and related topics (see for instance [2, 3, 21, 26, 28]).

Similarly to the relation between differential operator and integral operator as in the First Fundamental Theorem of Calculus, L. Guo and W. Keigher [23] introduced the notion of differential Rota-Baxter algebra which is a  $k$ -algebra  $R$  together with a differential operator  $D$  and a Rota-Baxter operator  $P$  such that  $DP = Id_R$ .

As we known, the free objects of various varieties of linear algebras play an important role. Sometimes, it is difficult to give a linear basis of a free algebra, for example, it is an open problem to find a linear basis of a free Jordan algebra. A linear basis of the free differential associative (resp. commutative and associative) Rota-Baxter algebra on a set was given by L. Guo and W. Keigher [23]. In this paper, we apply the Gröbner-Shirshov bases method to construct a free differential Lie Rota-Baxter algebra. Especially, we give a linear basis of a free differential Lie Rota-Baxter algebra on a set.

Gröbner bases and Gröbner-Shirshov bases have been proved to be very useful in different branches of mathematics, which were invented independently by A.I. Shirshov [32], H. Hironaka [24] and B. Buchberger [13] on different types of algebras. For more details on the Gröbner-Shirshov bases and their applications, see for instance the surveys [8, 10], the books [1, 11, 16, 18] and the papers [9, 17, 20, 27, 28].

The  $\Omega$ -algebra was introduced by A.G. Kurosh [25]. A differential Lie  $\Omega$ -algebra over a field  $k$  is a differential Lie algebra  $L$  with a set of multilinear operators  $\Omega$  on  $L$ . It is easy to see that a differential Lie Rota-Baxter algebra is a differential Lie  $\Omega$ -algebra with a single operator satisfying the Rota-Baxter relation.

The paper is organized as follows. In Section 2, we review the Gröbner-Shirshov bases theory for differential associative  $\Omega$ -algebras. In Section 3, we firstly construct a free differential Lie  $\Omega$ -algebra by the differential nonassociative Lyndon-Shirshov  $\Omega$ -words, which is a generalization of the classical nonassociative Lyndon-Shirshov words. Secondly, we establish the Gröbner-Shirshov bases theory for differential Lie  $\Omega$ -algebras. In Section 4, we obtain a Gröbner-Shirshov basis of a free  $\lambda$ -differential Lie Rota-Baxter algebra and then a linear basis of such an algebra is obtained by the Composition-Diamond lemma for differential Lie  $\Omega$ -algebras.

## 2 Gröbner-Shirshov bases for $\lambda$ -differential associative $\Omega$ -algebras

In this section, we briefly review the Gröbner-Shirshov bases theory for  $\lambda$ -differential associative  $\Omega$ -algebras, which can be found in [27].

### 2.1 Free $\lambda$ -differential associative $\Omega$ -algebras

Let  $D$  be a 1-ary operator and

$$\Omega := \bigcup_{m=1}^{\infty} \Omega_m,$$

where  $\Omega_m$  is a set of  $m$ -ary operators for any  $m \geq 1$ . For any set  $Y$ , we define the following notations:

$S(Y)$ : the set of all nonempty associative words on  $Y$ .

$Y^*$ : the set of all associative words on  $Y$  including the empty word 1.

$Y^{**}$ : the set of all nonassociative words on  $Y$ .

$$\Delta(Y) := \bigcup_{m=0}^{\infty} \{D^m(y) | y \in Y\}, \text{ where } D^0(y) = y, y \in Y.$$

$$\Omega(Y) := \bigcup_{m=1}^{\infty} \{\omega^{(m)}(y_1, y_2, \dots, y_m) | y_i \in Y, 1 \leq i \leq m, \omega^{(m)} \in \Omega_m\}.$$

Let  $X$  be a set. Define the differential associative and nonassociative  $\Omega$ -words on  $X$  as follows. For  $n = 0$ , define  $\langle D, \Omega; X \rangle_0 = S(\Delta(X))$ ,  $(D, \Omega; X)_0 = (\Delta(X))^{**}$ . For  $n > 0$ , define

$$\langle D, \Omega; X \rangle_n = S(\Delta(X \cup \Omega(\langle D, \Omega; X \rangle_{n-1}))),$$

$$(D, \Omega; X)_n = (\Delta(X \cup \Omega((D, \Omega; X)_{n-1})))^{**}.$$

Set

$$\langle D, \Omega; X \rangle = \bigcup_{n=0}^{\infty} \langle D, \Omega; X \rangle_n, \quad (D, \Omega; X) = \bigcup_{n=0}^{\infty} (D, \Omega; X)_n.$$

The elements of  $\langle D, \Omega; X \rangle$  (resp.  $(D, \Omega; X)$ ) are called differential associative (resp. nonassociative)  $\Omega$ -words on  $X$ . A differential associative  $\Omega$ -word  $u$  is called prime if  $u \in \Delta(X \cup \Omega(\langle D, \Omega; X \rangle))$ .

Let  $k$  be a field and  $\lambda \in k$ . A  $\lambda$ -differential associative  $\Omega$ -algebra over  $k$  is a  $\lambda$ -differential associative  $k$ -algebra  $R$  together with a set of multilinear operators  $\Omega$  on  $R$ .

Let  $DA\langle \Omega; X \rangle = k\langle D, \Omega; X \rangle$  be the semigroup algebra of  $\langle D, \Omega; X \rangle$ . Let  $u = u_1 u_2 \cdots u_t \in \langle D, \Omega; X \rangle$ , where each  $u_i$  is prime. If  $t = 1$ , i.e.  $u = D^i(u')$  for some  $i \geq 0, u' \in X \cup \Omega(\langle D, \Omega; X \rangle)$ , then we define  $D(u) = D^{i+1}(u')$ . If  $t > 1$ , then we recursively define

$$D(u) = D(u_1)(u_2 \cdots u_t) + u_1 D(u_2 \cdots u_t) + \lambda D(u_1) D(u_2 \cdots u_t).$$

Extend linearly  $D$  to  $DA\langle\Omega; X\rangle$ . For any  $\omega^{(m)} \in \Omega_m$ , define

$$\omega^{(m)} : \langle D, \Omega; X \rangle^m \rightarrow \langle D, \Omega; X \rangle, (u_1, u_2, \dots, u_m) \mapsto \omega^{(m)}(u_1, u_2, \dots, u_m)$$

and extend it linearly to  $DA\langle\Omega; X\rangle^m$ .

**Theorem 2.1** ([27])  *$(DA\langle\Omega; X\rangle, D, \Omega)$  is a free  $\lambda$ -differential associative  $\Omega$ -algebra on the set  $X$ .*

## 2.2 Composition-Diamond lemma for $\lambda$ -differential associative $\Omega$ -algebras

Let  $\star$  is a symbol, which is not in  $X$ . By a differential  $\star$ - $\Omega$ -word we mean any expression in  $\langle D, \Omega; X \cup \{\star\} \rangle$  with only one occurrence of  $\star$ . The set of all the differential  $\star$ - $\Omega$ -words on  $X$  is denoted by  $\langle D, \Omega; X \rangle^\star$ . Let  $\pi$  be a differential  $\star$ - $\Omega$ -word and  $s \in DA\langle\Omega; X\rangle$ . Then we call  $\pi|_s = \pi|_{\star \mapsto s}$  a differential  $s$ -word.

Let  $\deg(u)$  be the number of all occurrences of  $x \in X$ ,  $\omega \in \Omega$  and  $D$  in  $u$ . If  $u = u_1 u_2 \cdots u_m$ , where  $u_i$  is prime, then the breath of  $u$ , denoted by  $\text{bre}(u)$ , is defined to be the number  $m$ . Define

$$\text{wt}(u) = (\deg(u), \text{bre}(u), u_1, u_2, \dots, u_m).$$

Let  $X$  and  $\Omega$  be well-ordered sets and assume that  $\omega > D$  for any  $\omega \in \Omega$ . We define the Deg-lex order  $>_{Dl}$  on  $\langle D, \Omega; X \rangle$  as follows. For any  $u = u_1 u_2 \cdots u_n$  and  $v = v_1 v_2 \cdots v_m \in \langle D, \Omega; X \rangle$ , where  $u_i, v_j$  are prime, define

$$u >_{Dl} v \text{ if } \text{wt}(u) > \text{wt}(v) \text{ lexicographically,}$$

where if  $u_i = \omega(u_{i1}, u_{i2}, \dots, u_{it}), v_i = \theta(v_{i1}, v_{i2}, \dots, v_{il}), \omega, \theta \in \{D\} \cup \Omega$  and  $\deg(u_i) = \deg(v_i)$ , then  $u_i >_{Dl} v_i$  if

$$(\omega, u_{i1}, u_{i2}, \dots, u_{it}) > (\theta, v_{i1}, v_{i2}, \dots, v_{il}) \text{ lexicographically.}$$

It is easy to check that  $>_{Dl}$  is a well order on  $\langle D, \Omega; X \rangle$ . For any  $0 \neq f \in DA\langle\Omega; X\rangle$ , let  $\bar{f}$  be the leading term of  $f$  with respect to the order  $>_{Dl}$ . Let us denote  $lc(f)$  the coefficient of the leading term  $\bar{f}$  of  $f$ .

For  $1 \leq t \leq n$ , define

$$I_n^t = \{(i_1, i_2, \dots, i_n) \in \{0, 1\}^n | i_1 + i_2 + \dots + i_n = t\}.$$

**Lemma 2.2** ([27]) *If  $u = u_1 u_2 \cdots u_n \in \langle D, \Omega; X \rangle$ , where each  $u_i$  is prime, then*

$$\begin{aligned} D(u) &= \sum_{(i_1, i_2, \dots, i_n) \in I_n^1} D^{i_1}(u_1) D^{i_2}(u_2) \cdots D^{i_n}(u_n) \\ &+ \sum_{t=2}^n \sum_{(i_1, i_2, \dots, i_n) \in I_n^t} \lambda^{t-1} D^{i_1}(u_1) D^{i_2}(u_2) \cdots D^{i_n}(u_n). \end{aligned}$$

**Lemma 2.3** ([27]) Let  $u = u_1 u_2 \cdots u_n \in \langle D, \Omega; X \rangle$ , where each  $u_i$  is prime.

(a) If  $\lambda = 0$ , then  $\overline{D^i(u)} = D^i(u_1)u_2 \cdots u_n$  and  $lc(D^i(u)) = 1$ .

(b) If  $\lambda \neq 0$ , then  $\overline{D^i(u)} = D^i(u_1)D^i(u_2) \cdots D^i(u_n)$  and  $lc(D^i(u)) = \lambda^{(n-1)i}$ .

It follows that if  $u, v \in \langle D, \Omega; X \rangle$  and  $u >_{Dl} v$ , then  $\overline{D(u)} > \overline{D(v)}$ .

**Proposition 2.4** ([27]) For any  $u, v \in \langle D, \Omega; X \rangle$ ,  $\pi \in \langle D, \Omega; X \rangle^*$ , if  $u >_{Dl} v$ , then  $\overline{\pi|_u} >_{Dl} \overline{\pi|_v}$ .

If  $\overline{\pi|_s} = \pi|_{\overline{s}}$ , where  $s \in DA\langle \Omega; X \rangle$  and  $\pi \in \langle D, \Omega; X \rangle^*$ , then  $\pi|_s$  is called a normal differential  $s$ -word. Note that not each differential  $s$ -word is a normal differential  $s$ -word, for example, if  $u = D(x)P(D^2(\star))$  and  $s = xy$ , where  $P \in \Omega$ ,  $x, y \in X$ , then  $\pi|_s$  is not a normal differential  $s$ -word. However, if we take  $\pi' = D(x_1)P(\star)$ , then  $\pi|_s = \pi'|_{D^2(s)}$  and  $\pi'|_{D^2(s)}$  is a normal differential  $D^2(s)$ -word.

**Lemma 2.5** ([27]) For any differential  $s$ -word  $\pi|_s$ , there exist  $i \geq 0$  and  $\pi'$  such that  $\pi|_s = \pi'|_{D^i(s)}$  and  $\pi'|_{D^i(s)}$  is a normal differential  $D^i(s)$ -word.

Let  $f, g \in DA\langle \Omega; X \rangle$ . There are two kinds of compositions.

(i) If there exists a  $w = \overline{D^i(f)}a = \overline{bD^j(g)}$  for some  $a, b \in \langle D, \Omega; X \rangle$  such that  $bre(w) < bre(f) + bre(\bar{g})$ , then we call

$$(f, g)_w = lc(D^i(f))^{-1}D^i(f)a - lc(D^j(g))^{-1}bD^j(g)$$

the intersection composition of  $f$  and  $g$  with respect to the ambiguity  $w$ .

(ii) If there exists a  $\pi \in \langle D, \Omega; X \rangle^*$  such that  $w = \overline{D^i(f)} = \pi|_{\overline{D^j(g)}}$ , where

$\pi|_{\overline{D^j(g)}}$  is a normal differential  $D^j(g)$ -word, then we call

$$(f, g)_w = lc(D^i(f))^{-1}D^i(f) - lc(D^j(g))^{-1}\pi|_{\overline{D^j(g)}}$$

the inclusion composition of  $f$  and  $g$  with respect to the ambiguity  $w$ .

Let  $S$  be a subset of  $DA\langle \Omega; X \rangle$ . Then the composition  $(f, g)_w$  is called trivial modulo  $(S, w)$  if

$$(f, g)_w = \sum \alpha_i \pi_i|_{D^{l_i}(s_i)},$$

where each  $\alpha_i \in k$ ,  $\pi_i \in \langle D, \Omega; X \rangle^*$ ,  $s_i \in S$ ,  $\pi_i|_{D^{l_i}(s_i)}$  is a normal differential  $D^{l_i}(s_i)$ -word and  $\pi_i|_{\overline{D^{l_i}(s_i)}} <_{Dl} w$ . If this is the case, we write

$$(f, g)_w \equiv_{ass} 0 \mod(S, w).$$

In general, for any two polynomials  $p$  and  $q$ ,  $p \equiv_{ass} q \mod(S, w)$  means that  $p - q = \sum \alpha_i \pi_i|_{D^{l_i}(s_i)}$ , where each  $\alpha_i \in k$ ,  $\pi_i \in \langle D, \Omega; X \rangle^*$ ,  $s_i \in S$ ,  $\pi_i|_{D^{l_i}(s_i)}$  is a normal differential  $D^{l_i}(s_i)$ -word and  $\pi_i|_{\overline{D^{l_i}(s_i)}} <_{Dl} w$ .

A set  $S \subset DA\langle \Omega; X \rangle$  is called a Gröbner-Shirshov basis in  $DA\langle \Omega; X \rangle$  if any composition  $(f, g)_w$  of  $f, g \in S$  is trivial modulo  $(S, w)$ .

**Theorem 2.6** ([27], *Composition-Diamond lemma for differential associative  $\Omega$ -algebras*) Let  $S$  be a subset of  $DA\langle\Omega; X\rangle$ ,  $Id_{DA}(S)$  the ideal of  $DA\langle\Omega; X\rangle$  generated by  $S$  and  $>_{D^i}$  the Deg-lex order on  $\langle D, \Omega; X \rangle$  defined as before. Then the following statements are equivalent:

(i)  $S$  is a Gröbner-Shirshov basis in  $DA\langle\Omega; X\rangle$ .

(ii)  $f \in Id_{DA}(S) \Rightarrow \bar{f} = \pi|_{\frac{\quad}{D^i(s)}}$  for some  $\pi \in \langle D, \Omega; X \rangle^*$ ,  $s \in S$  and  $i \geq 0$ .

(iii) The set

$$Irr(S) = \left\{ w \in \langle D, \Omega; X \rangle \left| \begin{array}{l} w \neq \pi|_{\frac{\quad}{D^i(s)}}, s \in S, i \geq 0, \pi|_{D^i(s)} \text{ is} \\ \text{a normal differential } D^i(s)\text{-word} \end{array} \right. \right\}$$

is a linear basis of the differential associative  $\Omega$ -algebra  $DA\langle\Omega; X|S\rangle := DA\langle\Omega; X\rangle/Id_{DA}(S)$ .

### 3 Gröbner-Shirshov bases for $\lambda$ -differential Lie $\Omega$ -algebras

#### 3.1 Lyndon-Shirshov words

In this subsection, we review the concept and some properties of Lyndon-Shirshov words, which can be found in [7, 32].

For any  $u \in X^*$ , let us denote by  $\deg(u)$  the degree (length) of  $u$ . Let  $>$  be a well order on  $X$ . Define the lex-order  $>_{lex}$  and the deg-lex order  $>_{deg-lex}$  on  $X^*$  with respect to  $>$  by:

(i)  $1 >_{lex} u$  for any nonempty word  $u$ , and if  $u = x_i u'$  and  $v = x_j v'$ , where  $x_i, x_j \in X$ , then  $u >_{lex} v$  if  $x_i > x_j$ , or  $x_i = x_j$  and  $u' >_{lex} v'$  by induction.

(ii)  $u >_{deg-lex} v$  if  $\deg(u) > \deg(v)$ , or  $\deg(u) = \deg(v)$  and  $u >_{lex} v$ .

A nonempty associative word  $w$  is called an associative Lyndon-Shirshov word on  $X$ , if  $w = uv >_{lex} vu$  for any decomposition of  $w = uv$ , where  $1 \neq u, v \in X^*$ .

A nonassociative word  $(u) \in X^{**}$  is said to be a nonassociative Lyndon-Shirshov word on  $X$  with respect to the lex-order  $>_{lex}$ , if

- (a)  $u$  is an associative Lyndon-Shirshov word on  $X$ ;
- (b) if  $(u) = ((v)(w))$ , then both  $(v)$  and  $(w)$  are nonassociative Lyndon-Shirshov words on  $X$ ;
- (c) if  $(v) = ((v_1)(v_2))$ , then  $v_2 \leq_{lex} w$ .

Let  $ALSW(X)$  (resp.  $NLSW(X)$ ) denote the set of all the associative (resp. nonassociative) Lyndon-Shirshov words on  $X$  with respect to the lex-order  $>_{lex}$ . It is well known that for any  $u \in ALSW(X)$ , there exists a unique

Shirshov standard bracketing way  $[u]$  (see for instance [7]) on  $u$  such that  $[u] \in NLSW(X)$ . Then  $NLSW(X) = \{[u] | u \in ALSW(X)\}$ .

Let  $k\langle X \rangle$  be the free associative algebra on  $X$  over a field  $k$  and  $Lie(X)$  be the Lie subalgebra of  $k\langle X \rangle$  generated by  $X$  under the Lie bracket  $(uv) = uv - vu$ . It is well known that  $Lie(X)$  is a free Lie algebra on the set  $X$  and  $NLSW(X)$  is a linear basis of  $Lie(X)$ .

### 3.2 Differential Lyndon-Shirshov $\Omega$ -words

Let  $>_{Dl}$  be the Deg-lex order on  $\langle D, \Omega; X \rangle$  and  $\succ$  the restriction of  $>_{Dl}$  on  $\Delta(X \cup \Omega(\langle D, \Omega; X \rangle))$ . Define the differential Lyndon-Shirshov  $\Omega$ -words on the set  $X$  as follows.

For  $n = 0$ , let  $Z_0 := \Delta(X)$ . Define

$$ALSW(D, \Omega; X)_0 := ALSW(Z_0),$$

$$NLSW(D, \Omega; X)_0 := NLSW(Z_0) = \{[u] | u \in ALSW(D, \Omega; X)_0\}$$

with respect to the lex-order  $\succ_{lex}$  on  $(Z_0)^*$ , where  $[u]$  is the Shirshov standard bracketing way on  $u$ .

Assume that we have defined

$$ALSW(D, \Omega; X)_{n-1},$$

$$NLSW(D, \Omega; X)_{n-1} := \{[u] | u \in ALSW(D, \Omega; X)_{n-1}\}.$$

Let  $Z_n := \Delta(X \cup \Omega(ALSW(D, \Omega; X)_{n-1}))$ . Define

$$ALSW(D, \Omega; X)_n := ALSW(Z_n).$$

with respect to the lex-order  $\succ_{lex}$  on  $Z_n^*$ . For any  $u \in Z_n$ , define the bracketing way on  $u$  by

$$[u] := \begin{cases} u, & \text{if } u = D^i(x), x \in X, \\ D^i(\omega^{(m)}([u_1], [u_2], \dots, [u_m])), & \text{if } u = D^i(\omega^{(m)}(u_1, u_2, \dots, u_m)). \end{cases}$$

Let  $[Z_n] := \{[u] | u \in Z_n\}$ . Thus, the order  $\succ$  on  $Z_n$  induces an order on  $[Z_n]$  by  $[u] \succ [v]$  if  $u \succ v$  for any  $u, v \in Z_n$ . For any  $u = u_1 u_2 \dots u_t \in ALSW(D, \Omega; X)_n$ , where each  $u_i \in Z_n$ , we define

$$[u] := [[u_1][u_2] \dots [u_t]]$$

the Shirshov standard bracketing way on the word  $[u_1][u_2] \dots [u_t]$ , which means that  $[u]$  is a nonassociative Lyndon-Shirshov word on the set  $\{[u_1], [u_2], \dots, [u_t]\}$ . Define

$$NLSW(D, \Omega; X)_n := \{[u] | u \in ALSW(D, \Omega; X)_n\}.$$

It is easy to see that  $NLSW(D, \Omega; X)_n = NLSW([Z_n])$  with respect to the lex-order  $\succ_{lex}$  on  $[Z_n]^*$ .

Set

$$ALSW(D, \Omega; X) := \bigcup_{n=0}^{\infty} ALSW(D, \Omega; X)_n,$$

$$NLSW(D, \Omega; X) := \bigcup_{n=0}^{\infty} NLSW(D, \Omega; X)_n.$$

Then, we have

$$NLSW(D, \Omega; X) = \{[u] | u \in ALSW(D, \Omega; X)\}.$$

The elements of  $ALSW(D, \Omega; X)$  (resp.  $NLSW(D, \Omega; X)$ ) are called the differential associative (resp. nonassociative) Lyndon-Shirshov  $\Omega$ -words on the set  $X$ .

### 3.3 Free $\lambda$ -differential Lie $\Omega$ -algebras

In this subsection, we prove that the set  $NLSW(D, \Omega; X)$  of all differential nonassociative Lyndon-Shirshov  $\Omega$ -words on  $X$  forms a linear basis of the free  $\lambda$ -differential Lie  $\Omega$ -algebra on  $X$ .

A  $\lambda$ -differential Lie algebra is a Lie algebra  $L$  with a linear operator  $D : L \rightarrow L$  satisfying the differential relation

$$D([xy]) = [D(x)y] + [xD(y)] + \lambda[D(x)D(y)], x, y \in L.$$

A  $\lambda$ -differential Lie  $\Omega$ -algebra is a  $\lambda$ -differential Lie algebra  $L$  with a set of multilinear operators  $\Omega$  on  $L$ .

Let  $(R, \cdot, D, \Omega)$  is a  $\lambda$ -differential associative  $\Omega$ -algebra. Then it is easy to check that  $(R, [, ], D, \Omega)$  is a  $\lambda$ -differential Lie  $\Omega$ -algebra under the Lie bracket  $[a, a'] = a \cdot a' - a' \cdot a$ ,  $a, a' \in R$ .

Let  $DLie(\Omega; X)$  be the  $\lambda$ -differential Lie  $\Omega$ -subalgebra of  $DA\langle\Omega; X\rangle$  generated by  $X$  under the Lie bracket  $(uv) = uv - vu$ .

Similar to the proofs of Lemma 2.6 and Theorem 2.8 in [28], we have the following results.

**Lemma 3.1** *If  $u \in ALSW(D, \Omega; X)$ , then  $\overline{[u]} = u$  with respect to the order  $>_{DL}$  on  $\langle D, \Omega; X \rangle$ .*

**Theorem 3.2**  *$DLie(\Omega; X)$  is a free  $\lambda$ -differential Lie  $\Omega$ -algebra on the set  $X$  and  $NLSW(D, \Omega; X)$  is a linear basis of  $DLie(\Omega; X)$ .*

### 3.4 Composition-Diamond lemma for differential Lie $\Omega$ -algebras

In this subsection, we establish the Composition-Diamond lemma for differential Lie  $\Omega$ -algebras.



**Lemma 3.3** *Let  $\pi \in \langle D, \Omega; X \rangle^*$  and  $v, \pi|_v \in ALSW(D, \Omega; X)$ . Then there is a  $\pi' \in \langle D, \Omega; X \rangle^*$  and  $c \in \langle D, \Omega; X \rangle$  such that*

$$[\pi|_v] = [\pi'|_{[vc]}],$$

where  $c$  may be empty. Let

$$[\pi|_v]_v = [\pi'|_{[vc]}]_{[vc] \mapsto [\dots[[[v][c_1]][c_2]] \dots [c_m]]}$$

where  $c = c_1 c_2 \dots c_m$  with each  $c_i \in ALSW(D, \Omega; X)$  and  $c_t \preceq_{lex} c_{t+1}$ . Then,

$$[\pi|_v]_v = \pi|_{[v]} + \sum \alpha_i \pi_i|_{[v]},$$

where each  $\alpha_i \in k$  and  $\pi_i|_v <_{D^i} \pi|_v$ . It follows that  $\overline{[\pi|_v]_v} = \pi|_v$  with respect to the order  $>_{D^i}$ .

**Proof.** The proof is the same as the one of Lemma 3.2 in [28].  $\square$

Let  $0 \neq f \in D Lie(\Omega; X) \subseteq DA\langle \Omega; X \rangle$ . If  $\pi|_{\bar{f}} \in ALSW(D, \Omega; X)$ , then we call

$$[\pi|_f]_{\bar{f}} = [\pi|_{\bar{f}}]_{\bar{f} \mapsto f}$$

a special normal differential  $f$ -word.

**Corollary 3.4** *Let  $f \in D Lie(\Omega; X)$  and  $\pi|_{\bar{f}} \in ALSW(D, \Omega; X)$ . Then*

$$[\pi|_f]_{\bar{f}} = \pi|_f + \sum \alpha_i \pi_i|_f,$$

where each  $\alpha_i \in k$  and  $\pi_i|_{\bar{f}} <_{D^i} \pi|_{\bar{f}}$ .

Let  $f, g \in D Lie(\Omega; X)$ . There are two kinds of compositions.

- (i) If there exists a  $w = \overline{D^i(f)a} = \overline{bD^j(g)}$  for some  $a, b \in \langle D, \Omega; X \rangle$  such that  $bre(w) < bre(\bar{f}) + bre(\bar{g})$ , then we call

$$\langle f, g \rangle_w = lc(D^i(f))^{-1} [D^i(f)a]_{\overline{D^i(f)}} - lc(D^j(g))^{-1} [bD^j(g)]_{\overline{D^j(g)}}$$

the intersection composition of  $f$  and  $g$  with respect to the ambiguity  $w$ .

- (ii) If there exists a  $\pi \in \langle D, \Omega; X \rangle^*$  such that  $w = \overline{D^i(f)} = \pi|_{\overline{D^j(g)}}$ , where  $\pi|_{\overline{D^j(g)}}$  is a normal differential  $D^j(g)$ -word, then we call

$$\langle f, g \rangle_w = lc(D^i(f))^{-1} D^i(f) - lc(D^j(g))^{-1} [\pi|_{\overline{D^j(g)}}]_{\overline{D^j(g)}}$$

the inclusion composition of  $f$  and  $g$  with respect to the ambiguity  $w$ .

If  $S$  is a subset of  $DLie(\Omega; X)$ , then the composition  $\langle f, g \rangle_w$  is called trivial modulo  $(S, w)$  if

$$\langle f, g \rangle_w = \sum \alpha_i [\pi_i|_{D^{l_i}(s_i)}]_{\overline{D^{l_i}(s_i)}},$$

where each  $\alpha_i \in k$ ,  $s_i \in S$ ,  $[\pi_i|_{D^{l_i}(s_i)}]_{\overline{D^{l_i}(s_i)}}$  is a special normal differential  $D^{l_i}(s_i)$ -word and  $\pi_i|_{\overline{D^{l_i}(s_i)}} <_{Dl} w$ . If this is the case, then we write

$$\langle f, g \rangle_w \equiv 0 \mod(S, w).$$

In general, for any two polynomials  $p$  and  $q$ ,  $p \equiv q \mod(S, w)$  means that  $p - q = \sum \alpha_i [\pi_i|_{D^{l_i}(s_i)}]_{\overline{D^{l_i}(s_i)}}$ , where each  $\alpha_i \in k$ ,  $\pi_i \in \langle D, \Omega; X \rangle^*$ ,  $s_i \in S$ ,  $[\pi_i|_{D^{l_i}(s_i)}]_{\overline{D^{l_i}(s_i)}}$  is a normal differential  $D^{l_i}(s_i)$ -word and  $\pi_i|_{\overline{D^{l_i}(s_i)}} <_{Dl} w$ .

**Definition 3.5** A set  $S \subset DLie(\Omega; X)$  is called a Gröbner-Shirshov basis in  $DLie(\Omega; X)$  if any composition  $\langle f, g \rangle_w$  of  $f, g \in S$  is trivial modulo  $(S, w)$ .

**Lemma 3.6** Let  $f, g \in DLie(\Omega; X)$ . Then

$$\langle f, g \rangle_w - (f, g)_w \equiv_{ass} 0 \mod(\{f, g\}, w).$$

**Proof.** If  $\langle f, g \rangle_w$  and  $(f, g)_w$  are compositions of intersection, where  $w = \overline{D^i(f)a} = \overline{bD^j(g)}$ , then

$$\begin{aligned} & \langle f, g \rangle_w \\ &= lc(D^i(f))^{-1} [D^i(f)a]_{\overline{D^i(f)}} - lc(D^j(g))^{-1} [bD^j(g)]_{\overline{D^j(g)}} \\ &= lc(D^i(f))^{-1} D^i(f)b + \sum \alpha_i a_i D^i(f)a'_i - lc(D^j(g))^{-1} bD^j(g) - \sum \beta_j b_j D^j(g)b'_j \\ &= (f, g)_w + \sum \alpha_i a_i D^i(f)a'_i - \sum \beta_j b_j D^j(g)b'_j, \end{aligned}$$

where  $a_i \overline{D^i(f)} a'_i$ ,  $b_j \overline{D^j(g)} b'_j <_{Dl} w$ . It follows that

$$\langle f, g \rangle_w - (f, g)_w \equiv_{ass} 0 \mod(\{f, g\}, w).$$

If  $\langle f, g \rangle_w$  and  $(f, g)_w$  are compositions of inclusion, where  $w = \bar{f} = \pi|_{\overline{D^j(g)}}$ , then

$$\langle f, g \rangle_w = f - lc(D^j(g))^{-1} [\pi|_{D^j(g)}]_{\overline{D^j(g)}} = f - lc(D^j(g))^{-1} \pi|_{D^j(g)} - \sum \alpha_i \pi_i|_{D^j(g)},$$

where  $\pi_i|_{\overline{D^j(g)}} <_{Dl} w$ . It follows that

$$\langle f, g \rangle_w - (f, g)_w \equiv_{ass} 0 \mod(\{f, g\}, w).$$

The proof is complete.  $\square$

**Lemma 3.7** Let  $S \subset DLie(\Omega; X) \subset DA(\Omega; X)$ . Then the following two statements are equivalent:

(i)  $S$  is a Gröbner-Shirshov basis in  $DLie(\Omega; X)$ ,

(ii)  $S$  is a Gröbner-Shirshov basis in  $DA\langle\Omega; X\rangle$ .

**Proof.** (i)  $\implies$  (ii). Suppose that  $S$  is a Gröbner-Shirshov basis in  $DLie(\Omega; X)$ . Then, for any composition  $\langle f, g \rangle_w$ , we have

$$\langle f, g \rangle_w = \sum \alpha_i [\pi_i|_{D^{l_i}(s_i)}]_{D^{l_i}(s_i)},$$

where each  $\alpha_i \in k$ ,  $s_i \in S$ ,  $\pi_i|_{D^{l_i}(s_i)} <_{Dl} w$ . By Corollary 3.4, we have

$$\langle f, g \rangle_w = \sum \beta_t \pi_t|_{D^{l_t}(s_t)},$$

where each  $\beta_t \in k$ ,  $s_t \in S$ ,  $\pi_t|_{D^{l_t}(s_t)} <_{Dl} w$ . Therefore, by Lemma 3.6, we can obtain that

$$(f, g)_w \equiv_{ass} 0 \mod(S, w).$$

Thus,  $S$  is a Gröbner-Shirshov basis in  $DA\langle\Omega; X\rangle$ .

(ii)  $\implies$  (i). Assume that  $S$  is a Gröbner-Shirshov basis in  $DA\langle\Omega; X\rangle$ . Then, for any composition  $\langle f, g \rangle_w$  in  $S$ , we have  $\langle f, g \rangle_w \in DLie(\Omega; X)$  and  $\langle f, g \rangle_w \in Id_{DA}(S)$ . By Theorem 2.6,  $\overline{\langle f, g \rangle_w} = \pi_1|_{D^{i_1}(s_1)} \in ALSW(D, \Omega; X)$ . Let

$$h_1 = \langle f, g \rangle_w - \alpha_1 [\pi_1|_{D^{i_1}(s_1)}]_{D^{i_1}(s_1)},$$

where  $\alpha_1$  is the coefficient of  $\overline{\langle f, g \rangle_w}$ . Then,  $\overline{h_1} <_{Dl} \overline{\langle f, g \rangle_w}$ ,  $h_1 \in Id_{DA}(S)$  and  $h_1 \in DLie(\Omega; X)$ . Now, the result follows from induction on  $\overline{\langle f, g \rangle_w}$ .  $\square$

**Lemma 3.8** *Let  $S \subset DLie(\Omega; X)$  and*

$$Irr(S) = \{[w] | w \in ALSW(D, \Omega; X), w \neq \pi|_{D^i(s)}, s \in S, \pi \in \langle D, \Omega; X \rangle^*, i \geq 0\}.$$

*Then, for any  $h \in DLie(\Omega; X)$ ,  $h$  can be expressed by*

$$h = \sum \alpha_i [u_i] + \sum \beta_j [\pi_j|_{D^{l_j}(s_j)}]_{D^{l_j}(s_j)},$$

*where each  $\alpha_i, \beta_j \in k$ ,  $u_i \in ALSW(D, \Omega; X)$ ,  $u_i \leq_{Dl} \bar{h}$  and  $s_j \in S$ ,  $\pi_j|_{D^{l_j}(s_j)} \leq_{Dl} \bar{h}$ .*

**Proof.** By induction on  $\bar{h}$ , we can obtain the result.  $\square$

The following theorem is the Composition-Diamond lemma for differential Lie  $\Omega$ -algebras. It is a generalization of Shirshov's Composition lemma for Lie algebras [32], which was specialized to associative algebras by L.A. Bokut [6], see also G.M. Bergman [5] and B. Buchberger [12, 13].

**Theorem 3.9** (*Composition-Diamond lemma for differential Lie  $\Omega$ -algebras*)  
Let  $S \subset DLie(\Omega; X)$  be a nonempty set and  $Id_{DLie}(S)$  the ideal of  $DLie(\Omega; X)$  generated by  $S$ . Then the following statements are equivalent:

(I)  $S$  is a Gröbner-Shirshov basis in  $DLie(\Omega; X)$ .

(II)  $f \in Id_{DLie}(S) \Rightarrow \bar{f} = \pi | \frac{\quad}{D^i(s)} \in ALSW(D, \Omega; X)$  for some  $s \in S$ ,  $\pi \in \langle D, \Omega; X \rangle^*$  and  $i \geq 0$ .

(III) The set

$$Irr(S) = \{[w] | w \in ALSW(D, \Omega; X), w \neq \pi | \frac{\quad}{D^i(s)}, s \in S, \pi \in \langle D, \Omega; X \rangle^*, i \geq 0\}$$

is a linear basis of the  $\lambda$ -differential Lie  $\Omega$ -algebras  $DLie(\Omega; X|S)$ .

**Proof.** (I)  $\Rightarrow$  (II). Since  $f \in Id_{DLie}(S) \subseteq Id_{DA}(S)$ , by Lemma 3.7 and Theorem 2.6, we have  $\bar{f} = \pi | \frac{\quad}{D^i(s)}$  for some  $s \in S$ ,  $\pi \in \langle D, \Omega; X \rangle^*$  and  $i \geq 0$ .

(II)  $\Rightarrow$  (III). Suppose that  $\sum \alpha_i [u_i] = 0$  in  $DLie(\Omega; X|S)$ , where each  $[u_i] \in Irr(S)$  and  $u_i >_{DI} u_{i+1}$ . That is,  $\sum \alpha_i [u_i] \in Id_{DLie}(S)$ . Then each  $\alpha_i$  must be 0. Otherwise, say  $\alpha_1 \neq 0$ , since  $\sum \alpha_i [u_i] = u_1$  and by (II), we have  $[u_1] \in Id_{DLie}(S)$ , a contradiction. Therefore,  $Irr(S)$  is linear independent. By Lemma 3.8,  $Irr(S)$  is a linear basis of  $DLie(\Omega; X|S) = DLie(\Omega; X)/Id_{DLie}(S)$ .

(III)  $\Rightarrow$  (I). For any composition  $\langle f, g \rangle_w$  with  $f, g \in S$ , we have  $\langle f, g \rangle_w \in Id_{DLie}(S)$ . Then, by (III) and by Lemma 3.8,

$$\langle f, g \rangle_w = \sum \beta_j [\pi | \frac{\quad}{D^{l_j}(s_j)}] \frac{\quad}{D^{l_j}(s_j)}$$

where each  $\beta_j \in k$ ,  $\pi | \frac{\quad}{D^{l_j}(s_j)} <_{DI} w$ . This proves that  $S$  is a Gröbner-Shirshov basis in  $DLie(\Omega; X)$ .  $\square$

## 4 Free $\lambda$ -differential Rota-Baxter Lie algebras

In this section, by using Theorem 3.9 we give a Gröbner-Shirshov basis of a free  $\lambda$ -differential Rota-Baxter Lie algebra on a set  $X$  and then a linear basis of such an algebra is obtained.

### 4.1 Gröbner-Shirshov bases for free $\lambda$ -differential Lie Rota-Baxter algebras

Let  $k$  be a field and  $\lambda \in k$ . A differential Lie Rota-Baxter algebra of weight  $\lambda$ , called also  $\lambda$ -differential Lie Rota-Baxter algebra, is a Lie algebra  $L$  with two linear operators  $P, D : L \rightarrow L$  such that for any  $x, y \in L$ ,

(a) (Rota-Baxter relation)  $[P(x)P(y)] = P([xP(y)]) + P([P(x)y]) + \lambda P([xy]);$

- (b) (differential relation)  $D([xy]) = [D(x)y] + [xD(y)] + \lambda[D(x)D(y)];$   
(c) (section relation)  $D(P(x)) = x.$

It is easy to see that any  $\lambda$ -differential Lie Rota-Baxter algebra is a  $\lambda$ -differential Lie  $\{P\}$ -algebra satisfying the relations (a) and (c).

Let  $DLie(\{P\}; X)$  be the free  $\lambda$ -differential Lie  $\{P\}$ -algebra on the set  $X$  and write

$$g(u) := D(P([u])) - [u],$$

$f(u, v) := [P([u])P([v])] - P([([u]P([v]))]) - P([P([u])[v]]) - \lambda P([u][v]), u >_{Dl} v,$   
where  $u, v \in ALSW(D, \{P\}; X)$ . Set

$$S = \{f(u, v), g(w) | u, v, w \in ALSW(D, \{P\}; X), u >_{Dl} v\}.$$

It is clear that  $DRBL(X) := DLie(\{P\}; X|S)$  is a free  $\lambda$ -differential Lie Rota-Baxter algebra on  $X$ .

For any  $f \in DLie(\{P\}; X)$ , let us denote  $r(f) := f - lc(f)[\overline{f}]$ .

**Lemma 4.1** *The set  $S_1 := \{D(P([u])) - [u] | u \in ALSW(D, \{P\}; X)\}$  is a Gröbner-Shirshov basis in  $DLie(\{P\}; X)$ .*

**Proof.** It is easy to check that  $S_1$  is a Gröbner-Shirshov basis in  $DLie(\{P\}; X)$ .  
 $\square$

**Lemma 4.2** *Let  $u, v \in ALSW(D, \{P\}; X)$  and  $u >_{Dl} v$ .*

(a) *If  $\lambda \neq 0$  and  $j > 0$ , then*

$$D^j(f(u, v)) \equiv \lambda^j(\overline{D^j(f(u, v))}) - (D^{j-1}([u])D^{j-1}([v])) \mod(S_1, \overline{D^j(f(u, v))}).$$

(b) *If  $\lambda = 0$  and  $j > 0$ , then*

$$D^j(f(u, v)) \equiv \overline{D^j(f(u, v))} - (D^{j-1}([u])P([v])) \mod(S_1, \overline{D^j(f(u, v))}).$$

**Proof.** (a) The proof is by induction on  $j$ . For  $j = 1$ , we have

$$\begin{aligned} & D(f(u, v)) \\ &= D((P([u])P([v])) - D(P([([u]P([v]))]) - D(P([P([u])[v]])) - \lambda D(P([u][v]))) \\ &\equiv \lambda(D(P([u])D(P([v]))) - \lambda D(P([([u]P([v]))])) \\ &\equiv \lambda(\overline{D(f(u, v))}) - ([u][v]) \mod(S_1, \overline{D(f(u, v))}). \end{aligned}$$

Assume that the result is true for  $j - 1, j \geq 2$ , i.e.

$$\begin{aligned} D^{j-1}(f(u, v)) &= \lambda^{j-1}(D^{j-1}(P([u])D^{j-1}(P([v]))) \\ &\quad - \lambda^{j-1}(D^{j-2}([u])D^{j-2}([v])) + \sum \alpha_i [\pi_i]_{D^{t_i}(s_i)} \overline{D^{t_i}(s_i)}], \end{aligned}$$

where each  $\alpha_i \in k$ ,  $s_i \in S_1$ ,  $\pi_i|_{\overline{D^{t_i}(s_i)}} <_{D^l} D^{j-1}(P(u))D^{j-1}(P(v))$ . Since  $S_1$  is a Gröbner-Shirshov basis in  $DLie(\{P\}; X)$ ,

$$D(\sum \alpha_i [\pi_i|_{\overline{D^{t_i}(s_i)}}]) = \sum \beta_l [\sigma_l|_{\overline{D^{n_l}(s_l)}}],$$

where each  $\beta_l \in k$ ,  $s_l \in S_1$ ,  $[\sigma_l|_{\overline{D^{n_l}(s_l)}}]$  is a special normal differential  $D^{k_l}(s_l)$ -word. By Lemma 2.3,

$$\begin{aligned} \overline{[\sigma_l|_{\overline{D^{n_l}(s_l)}}]} &= \sigma_l|_{\overline{D^{n_l}(s_l)}} \\ &<_{D^l} \overline{D((D^{j-1}(P([u]))D^{j-1}(P([v]))))} = D^j(P(u))D^j(P(v)). \end{aligned}$$

Thus, we have

$$\begin{aligned} &D^j(f(u, v)) \\ &= D(D^{j-1}(f(u, v))) \\ &\equiv \lambda^{j-1}D((D^{j-1}(P([u]))D^{j-1}(P([v])))) - \lambda^{j-1}D((D^{j-2}([u])D^{j-2}([v]))) \\ &\equiv \lambda^j(D^j(P([u]))D^j(P([v]))) + \lambda^{j-1}(D^j(P([u]))D^{j-1}(P([v]))) \\ &\quad + \lambda^{j-1}(D^{j-1}(P([u]))D^j(P([v]))) - \lambda^j(D^{j-1}([u])D^{j-1}([v])) \\ &\quad - \lambda^{j-1}(D^{j-1}([u])D^{j-2}([v])) - \lambda^{j-1}(D^{j-2}([u])D^{j-1}([v])) \\ &\equiv \lambda^j(D^j(P([u]))D^j(P([v]))) + \lambda^{j-1}(D^{j-1}([u])D^{j-2}([v])) \\ &\quad + \lambda^{j-1}(D^{j-2}([u])D^{j-1}([v])) - \lambda^j(D^{j-1}([u])D^{j-1}([v])) \\ &\quad - \lambda^{j-1}(D^{j-1}([u])D^{j-2}([v])) - \lambda^{j-1}(D^{j-2}([u])D^{j-1}([v])) \\ &\equiv \lambda^j(D^j(P([u]))D^j(P([v])) - \lambda^j(D^{j-1}([u])D^{j-1}([v])) \\ &\equiv \lambda^j(\overline{D^j(f(u, v))}) - (D^{j-1}([u])D^{j-1}([v])) \mod(S_1, \overline{D^j(f(u, v))}). \end{aligned}$$

(b) The proof is similar to Case (a).  $\square$

**Theorem 4.3** *With the order  $>_{D^l}$  on  $\langle D, \{P\}; X \rangle$  defined as before, the set  $S$  is a Gröbner-Shirshov basis in  $DLie(\{P\}; X)$ .*

**Proof.** There are two cases  $\lambda \neq 0$  and  $\lambda = 0$  to consider.

Case 1. For  $\lambda \neq 0$ , all possible compositions of the polynomials in  $S$  are list as below:

$$\begin{aligned} &\langle g(\pi|_{\overline{D^i(D(P(v)))}}), g(v) \rangle_{w_1}, \quad w_1 = D^j(D(P(\pi|_{\overline{D^i(D(P(v)))}}))), \\ &\langle g(\pi|_{\overline{D^i(P(u))D^i(P(v))}}), f(u, v) \rangle_{w_2}, \quad w_2 = D^j(D(P(\pi|_{\overline{D^i(P(u))D^i(P(v))}}))), \\ &\langle f(u, v), g(v) \rangle_{w_3}, \quad w_3 = D^l(P(u))D^l(P(v)), \quad l > 0, \\ &\langle f(u, v), g(u) \rangle_{w_4}, \quad w_4 = D^l(P(u))D^l(P(v)), \quad l > 0, \\ &\langle f(\pi|_{\overline{D^i(D(P(u)))}}), v, g(u) \rangle_{w_5}, \quad w_5 = D^j(P(\pi|_{\overline{D^i(D(P(u)))}}))D^j(P(v)), \\ &\langle f(u, \pi|_{\overline{D^i(D(P(v)))}}), g(v) \rangle_{w_6}, \quad w_6 = D^j(P(u))D^j(P(\pi|_{\overline{D^i(D(P(v)))}}))), \\ &\langle f(u, v), f(v, w) \rangle_{w_7}, \quad w_7 = D^j(P(u))D^j(P(v))D^j(P(w)), \end{aligned}$$

$$\begin{aligned} &\langle f(\pi|_{D^i(P(u))D^i(P(v))}, w), f(u, v) \rangle_{w_8}, \quad w_8 = D^j(P(\pi|_{D^i(P(u))D^i(P(w))}))D^j(P(w)), \\ &\langle f(u, \pi|_{D^i(P(v))D^i(P(w))}), f(v, w) \rangle_{w_9}, \quad w_9 = D^j(P(u))D^j(P(\pi|_{D^i(P(v))D^i(P(w))})), \end{aligned}$$

where  $i, j \geq 0$ .

We check that all the compositions in  $S$  are trivial. Here, we just check one composition as example.

If  $j > 0$ , then by Lemma 4.2, we have

$$\begin{aligned} &\langle f(\pi|_{D^i(P(u))D^i(P(v))}, w), f(u, v) \rangle_{w_8} \\ &= \lambda^{-j} D^j(f(\pi|_{D^i(P(u))D^i(P(v))}, w)) - \lambda^{-i} [D^j(P(\pi|_{D^i(f(u,v))}))D^j(P(w))] \frac{1}{D^i(f(u,v))} \\ &\equiv -(D^{j-1}([\pi|_{D^i(P(u))D^i(P(v))}])D^{j-1}([w])) - \lambda^{-i} (D^j(P(r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))})))D^j(P([w])) \\ &\equiv \lambda^{-i} (D^{j-1}(r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))})))D^{j-1}([w])) - \lambda^{-i} (D^{j-1}(r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))})))D^{j-1}([w])) \\ &\equiv 0 \text{ mod}(S, w_8). \end{aligned}$$

If  $j = 0$ , then

$$\begin{aligned} &\langle f(\pi|_{D^i(P(u))D^i(P(v))}, w), f(u, v) \rangle_{w_8} \\ &= f(\pi|_{D^i(P(u))D^i(P(v))}, w) - \lambda^{-i} [P(\pi|_{D^i(f(u,v))})P(w)] \frac{1}{D^i(f(u,v))} \\ &\equiv -P(([\pi|_{D^i(P([u]))D^i(P([v]))}]) [w])) - P(([\pi|_{D^i(P(u))D^i(P(v))}] P([w]))) \\ &\quad - \lambda P(([\pi|_{D^i(P(u))D^i(P(v))}] [w])) - \lambda^{-i} (P(r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))})))P([w])) \\ &\equiv \lambda^{-i} P((P(r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))}))) [w])) + \lambda^{-i} P((r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))}))) \\ &\quad \lambda^{-i+1} P((r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))}))) [w])) - \lambda^{-i} P((P(r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))}))) [w])) \\ &\quad - \lambda^{-i} P((r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))})))) - \lambda^{-i+1} P((r([\pi|_{D^i(f(u,v))}] \frac{1}{D^i(f(u,v))}))) [w])) \\ &\equiv 0 \text{ mod}(S, w_8). \end{aligned}$$

Case 2. For  $\lambda = 0$ , all possible compositions of the polynomials in  $S$  are list as below:

$$\begin{aligned} &\langle g(\pi|_{D^i(D(P(v)))}, g(v) \rangle_{w_1}, \quad w_1 = D^j(D(P(\pi|_{D^i(D(P(v))}))), \\ &\langle g(\pi|_{D^i(P(u))P(v)}), f(u, v) \rangle_{w_2}, \quad w_2 = D^j(D(P(\pi|_{D^i(P(u))P(v)}))), \\ &\langle f(u, v), g(u) \rangle_{w_3}, \quad w_3 = D^l(P(u))P(v), \quad l > 0, \\ &\langle f(\pi|_{D^i(D(P(u)))}, v), g(u) \rangle_{w_4}, \quad w_4 = D^j(P(\pi|_{D^i(D(P(u))})))P(v), \\ &\langle f(u, \pi|_{D^i(D(P(v)))}, g(v) \rangle_{w_5}, \quad w_5 = D^j(P(u))P(\pi|_{D^i(D(P(v))})), \\ &\langle f(u, v), f(v, w) \rangle_{w_6}, \quad w_6 = D^j(P(u))P(v)P(w), \\ &\langle f(\pi|_{D^i(P(u))P(v)}, w), f(u, v) \rangle_{w_7}, \quad w_7 = D^j(P(\pi|_{D^i(P(u))P(v)})))P(w), \\ &\langle f(u, \pi|_{D^i(P(v))P(w)}, f(v, w) \rangle_{w_8}, \quad w_8 = D^j(P(u))P(\pi|_{D^i(P(v))P(w)})), \end{aligned}$$

where  $i, j \geq 0$ . We check that all the compositions in  $S$  are trivial. The proof is similar to Case 1.  $\square$

## 4.2 A linear basis of a free $\lambda$ -differential Lie Rota-Baxter algebra

In this subsection, by Theorems 3.9 and 4.3, we obtain a linear basis of the free  $\lambda$ -differential Lie Rota-Baxter algebra on the set  $X$ .

For  $n = 0$ , define  $\langle \{P\}; \Delta(X) \rangle_0 := S(\Delta(X))$  and  $(\{P\}; \Delta(X))_0 := (\Delta(X))^{**}$ . For  $n > 0$ , define

$$\begin{aligned}\langle \{P\}; \Delta(X) \rangle_n &:= S(\Delta(X) \cup P(\langle \{P\}; \Delta(X) \rangle_{n-1})), \\ (\{P\}; \Delta(X))_n &:= (\Delta(X) \cup P(\{P\}; \Delta(X))_{n-1})^{**}.\end{aligned}$$

Set

$$\langle \{P\}; \Delta(X) \rangle := \bigcup_{n=0}^{\infty} \langle \{P\}; \Delta(X) \rangle_n, \quad (\{P\}; \Delta(X)) := \bigcup_{n=0}^{\infty} (\{P\}; \Delta(X))_n.$$

Let  $\star$  is a symbol, which is not in  $X$ . By a  $\star$ - $\Omega$ -word on  $\Delta(X)$ , we mean any expression in  $\langle \{P\}; \Delta(X) \cup \{\star\} \rangle$  with only one occurrence of  $\star$ . We will denote by  $\langle \{P\}; \Delta(X) \rangle^*$  the set of all the  $\star$ - $\Omega$ -words on  $\Delta(X)$ . Let  $\pi$  be a  $\star$ - $\Omega$ -word on  $\Delta(X)$  and  $u \in \langle \{P\}; \Delta(X) \rangle$ . Let us denote  $\pi|_u = \pi|_{\star \mapsto u}$ , i.e.  $\star$  is replaced by  $u$ .

It is easy to see that  $\langle \{P\}; \Delta(X) \rangle \subseteq \langle D, \{P\}; X \rangle$ . We also use the order  $>_{DI}$  on  $\langle \{P\}; \Delta(X) \rangle$  and  $\succ$  on  $\Delta(X) \cup P(\langle \{P\}; \Delta(X) \rangle)$ .

For  $n = 0$ , let  $Y_0 = \Delta(X)$ . Define

$$ALSW(\{P\}; \Delta(X))_0 := ALSW(Y_0),$$

$$NLSW(\{P\}; \Delta(X))_0 := NLSW(Y_0) = \{[u] | u \in ALSW(\{P\}; \Delta(X))_0\}$$

with respect to the lex-order  $\succ_{lex}$ .

Assume that we have defined

$$ALSW(\{P\}; \Delta(X))_{n-1},$$

$$NLSW(\{P\}; \Delta(X))_{n-1} = \{[u] | u \in ALSW(\{P\}; \Delta(X))_{n-1}\}.$$

Let  $Y_n := \Delta(X) \cup P(ALSW(\{P\}; \Delta(X))_{n-1})$ . Define

$$ALSW(\{P\}; \Delta(X))_n := ALSW(Y_n).$$

For any  $u \in Y_n$ , define the bracketing way on  $u$  by

$$[u] := \begin{cases} u, & \text{if } u \in \Delta(X), \\ P([u_1]), & \text{if } u = P(u_1). \end{cases}$$

Let  $[Y_n] := \{[u] | u \in Y_n\}$ . Therefore, the order  $\succ$  on  $Y_n$  induces an order on  $[Y_n]$  by  $[u] \succ [v]$  if  $u \succ v$  for any  $u, v \in Y_n$ . For any  $u = u_1 u_2 \cdots u_m \in ALSW(\{P\}; \Delta(X))_n$ , where each  $u_i \in Y_n$ , let us denote

$$[u] := [[u_1][u_2] \cdots [u_m]]$$



the nonassociative Lyndon-Shirshov word on  $\{[u_1], [u_2], \dots, [u_m]\}$  with respect to the lex-order  $\succ_{lex}$ .

Define

$$NLSW(\{P\}; \Delta(X))_n := \{[u] | u \in ALSW(\{P\}; \Delta(X))_n\}.$$

It is easy to see that  $NLSW(\{P\}; \Delta(X))_n = NLSW([Y_n])$ . Define

$$ALSW(\{P\}; \Delta(X)) := \bigcup_{n=0}^{\infty} ALSW(\{P\}; \Delta(X))_n,$$

$$NLSW(\{P\}; \Delta(X)) := \bigcup_{n=0}^{\infty} NLSW(\{P\}; \Delta(X))_n.$$

Therefore,

$$NLSW(\{P\}; \Delta(X)) = \{[u] | u \in ALSW(\{P\}; \Delta(X))\}.$$

By Theorems 3.9 and 4.3, we have the following theorem.

**Theorem 4.4** *The set*

$$Irr(S) = \left\{ [w] \in NLSW(\{P\}; \Delta(X)) \left| \begin{array}{l} w \neq \pi|_{P(u)P(v)}, \quad \pi \in \langle \{P\}; \Delta(X) \rangle^* \\ u, v \in ALSW(\{P\}; \Delta(X)), u >_{Dl} v \end{array} \right. \right\}$$

*is a linear basis of the free  $\lambda$ -differential Lie Rota-Baxter algebra  $DRBL(X)$  on  $X$ .*

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